Relativistic Modeling Capabilities In PERSEUS Extended MHD Simulation Code For HED Plasmas

Nathaniel D. Hamlin, Charles E. Seyler

Laboratory of Plasma Studies, Cornell University

40% of support: DOE cooperative agreement DE-NA-0001836
60% of support: DOE cooperative agreement DE-FOA-0001153
Outline

• Motivation
• Two-fluid equations
• Generalized Ohm’s law
• Primitive variable recovery
• Recovery of non-relativistic results: planar MRT
• Validation against PIC result: laser penetration problem
• Laser-foil interaction
Motivation for Relativistic Modeling

• Code must remain self-consistent when particles (usually electrons) become relativistic (kinetic and/or thermal energy comparable to rest energy $mc^2$).

• Can occur in many HED phenomena (laser-plasma interactions, X-pinches, electrode surface plasmas).
Two-fluid Equations

- Maxwell: \( \frac{\partial B}{\partial t} = - \nabla \times E, \quad \frac{\partial E}{\partial t} = c^2 \nabla \times B - \frac{J}{\varepsilon_0} \)

- Continuity: \( \frac{\partial N_{ei}}{\partial t} + \nabla \cdot (N_{ei} \mathbf{v}_{ei}) = 0, \quad N_{ei} = \gamma_{ei} n_{ei} \)
  \[ \gamma_{ei} = \frac{1}{\sqrt{1 - v_{ei}^2/c^2}} \]

- Momentum: \( \frac{\partial \Pi_{ei}}{\partial t} + \nabla \cdot (\Pi_{ei} \mathbf{v}_{ei} + P_{ei} \mathbf{I}) = q_{ei} N_{ei} (\mathbf{E} + \mathbf{v}_{ei} \times \mathbf{B}) + R_f \)
  \[ \Pi_{ei} = N_{ei} M_{ei} \gamma_{ei} \mathbf{v}_{ei}, \quad M_{ei} = m_{ei} \left[ 1 + \left( \frac{\Gamma}{\Gamma - 1} \right) \frac{P_{ei} \gamma_{ei}}{N_{ei} m_{ei} c^2} \right], \quad q_{ei} = \{-e, Ze\} \]

- Energy: \( \frac{\partial \epsilon_{ei}}{\partial t} + \nabla \cdot \left[ \mathbf{v}_{ei} (\epsilon_{ei} + P_{ei}) \right] = q_{ei} N_{ei} \mathbf{E} \cdot \mathbf{v}_{ei} + R_{\text{coll}} \)
  \[ \epsilon_{ei} = (\gamma_{ei} - 1) N_{ei} m_{ei} c^2 + P_{ei} \left( \frac{\Gamma}{\Gamma - 1} \gamma_{ei}^2 - 1 \right) \]
  \[ \gamma_{ei} = \frac{1}{2} m_{ei} v_{ei}^2 + \frac{P_{ei}}{\Gamma - 1} \]
  \[ \text{Non-relativistic:} \quad \epsilon_{ei} \to \frac{1}{2} m_{ei} v_{ei}^2 + \frac{P_{ei}}{\Gamma - 1} \]
Generalized Ohm’s Law

\[ \frac{\partial J_c}{\partial t} + \nabla \cdot F_J = \]
\[ \frac{N_e e^2}{m_e} \left( E + \frac{Z N_i}{N_e} \mathbf{v}_i \times B - \frac{J}{N_e e} \times B - \frac{\eta}{\gamma_e} J \right) + O \left( \frac{m_e}{m_i} \right) \]

- \( J_c = \frac{Z e}{m_i} \Pi_i - \frac{e}{m_e} \Pi_e \)
- \( F_J = \frac{Z e}{m_i} (\Pi_i \mathbf{v}_i + P_i I) - \frac{e}{m_e} (\Pi_e \mathbf{v}_e + P_e I) \)
- \( J = Z e N_i \mathbf{v}_i - e N_e \mathbf{v}_e \)

Charge conservation:

- \( \frac{\partial \rho_c}{\partial t} + \nabla \cdot J = 0, \quad \rho_c = Z e N_i - e N_e \)
Primitive Variable Recovery

- Two-fluid equations have form \( \frac{\partial Q}{\partial t} + \nabla \cdot F = S \).
- At each time step, must recover primitive variables \((n_j, v_j, \text{and } P_j)\) from conserved \(Q\) \((N_j, \Pi_j, \text{and } \epsilon_j)\) in order compute fluxes \(F\) and sources \(S\).
- Primitive variable recovery is done for electrons and ions separately.
- Equations to solve:
  - \( N_j = \gamma_j n_j \)
  - \( \Pi_j = \gamma_j v_j N_j m_j \left(1 + \frac{C_P P_j \gamma_j}{N_j m_j c^2}\right) \), \( C_P = \frac{\Gamma}{\Gamma - 1} \)
  - \( \epsilon_j = (\gamma_j - 1) N_j m_j c^2 + P_j (C_P \gamma_j^2 - 1) \)
- Combine into single equation in one variable (either \(P_j\) or \(\gamma_j^2 - 1\)).
- Solution with positive \(P_j\) exists only if
  \[
  \left(\frac{\epsilon_j}{N_j m_j c^2} + 1\right)^2 > \frac{\Pi_j^2}{N_j^2 m_j^2 c^2} \quad \text{and} \quad \frac{\epsilon_j}{N_j m_j c^2} + 1 > \gamma_j .
  \]
Relativistic vs Non-relativistic Comparison: Planar Foil Ablation

\( B_{\text{edge}} \)

\( J_y \)  
foil  
plasma ablation

Driving field

100 ns
Planar Foil Ablation

Log Ion Density (m$^{-3}$), 58 ns (0.58 rise times)

Non-relativistic

Relativistic
Planar Foil Ablation

Log Ion Density (m$^{-3}$), 125 ns (1.25 rise times)

Currently being applied to non-rel. 3D cylindrical MRT for benchmarking against Hall MHD PERSEUS code.
Laser Penetration into Over-dense Gas: Motivation

- Applicable to laser-driven capsule fusion by **fast ignition** (FI).
  - (1) Pre-compression of fuel core by laser-driven implosion and shock-heating
  - (2) Short, intense laser pulse penetrates over-dense plasma around core
  - (3) Laser pulse delivers relativistic electron beam which heats core to ignition temperatures

**Fast ignition concept**

- Matsuoka et al., Plasma Physics and Controlled Fusion, **50**, 2008
- Lei et al., Phys. Plasmas, **16**, 2009
Laser Penetration: Setup


\[ S = 10^{24} \text{ W/m}^2 \]
\[ \lambda = 1 \mu\text{m} \]
\[ \omega = 1.9 \times 10^{15} \text{ s}^{-1} \]

\[ \Delta x = \Delta y = \frac{\lambda}{30} \]
\[ L_x = 32\lambda, \quad L_y = 20\lambda \]
\[ \Delta t = \frac{0.22}{\omega_{pe,max}} \sim 0.004 \text{ to } 0.02 \text{ fs} \]
\[ T = \frac{2\pi}{\omega} \sim 160\Delta t \text{ to } 800\Delta t \]

Gaussian, width \( 6\lambda \)
\[ \omega = \omega_{pe} \text{ at } n_c = 1.12 \times 10^{27} \text{ m}^{-3} \]
GOL vs Pure two-fluid

• Pure two-fluid requires $\omega_{pe}\Delta t < 1$. **Explicit** advancement:
  - $E^{n+1} = E^n - \tilde{c}^2 \Delta t [\nabla \times B^n - J^n]$
  - $\Pi_e^{n+1} = \Pi_e^n - \Delta t [\nabla \cdot F(\Pi_e^n) - S(\Pi_e^n)]$

• GOL two-fluid relaxes to correct solution when $\omega_{pe}\Delta t > 1$. Solve local **implicit** system for $E^{n+1}, J_c^{n+1}$:
  - $E^{n+1} = E^n - \tilde{c}^2 \Delta t [\nabla \times B^n - J^{n+1}]$
  - $J_c^{n+1} = J_c^n - \Delta t \left[\nabla \cdot F(J_c^n) - N_e^n \frac{L_0}{\lambda_e^2} \left(E^{n+1} + \frac{Z N_i^n}{N_e^n} \nu_i^n \times B^n - \frac{\lambda_i}{L_0} \frac{N_i^n}{N_e^n} \times B^n - \frac{\eta^n}{\gamma_e^n} J^{n+1}\right)\right]$
  - $J^{n+1} = \frac{m_e}{M_e^n} \frac{J_c^{n+1}}{\gamma_e^n} + J_a$
Pure two-fluid, 333 fs  
GOL two-fluid, 333 fs

Cycle-averaged out-of-plane B-field over amplitude

Initial vacuum-plasma interface

Instantaneous out-of-plane B-field (T)

PIC simulation, Pukhov 1998

Intensity (10^{19} \text{ W/cm}^2)
Electron horizontal velocity (m/s)

Initial vacuum-plasma interface

Ion density over critical density

Pure two-fluid, 333 fs

GOL two-fluid, 333 fs

PIC simulation, Pukhov 1998
Penetration of laser into plasma: RIT

- Relativistic Induced Transparency (RIT)
- In fluid model, index of refraction $\eta_r$ can increase due to two relativistic corrections:
  - **Directional** electron motion (Lorentz factor $\gamma_e$)
  - **Random thermal** electron motion
    - mass correction $m_e/M_e = 1 + C_p T_e/(m_e v_e^2)$
- If electrons are sufficiently relativistic, then $n_{refr}$ is rendered positive thus enabling penetration, i.e.

$$0 < n_{refr}^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\gamma_e \omega^2} \left( \frac{m_e}{M_e} \right) = 1 - \frac{n_e}{\gamma_e n_c} \left( \frac{m_e}{M_e} \right)$$

- Electron heating (as opposed to directional motion) is primarily responsible:
Total energy composition vs time

- Black = total energy
- Red, brown = energy in electric, magnetic fields
- Blue = thermal (dashed), kinetic (solid) \textbf{electron} energy
- Green = thermal (dashed), kinetic (solid) \textbf{ion} energy
Laser-foil Setup

Solid Al foil (143 nm thick)

\[ S = 10^{24} \text{ W/m}^2 \]
\[ \lambda = 1 \mu\text{m} \]
\[ \omega = 1.9 \times 10^{15} \text{ s}^{-1} \]

\[ \Delta t = \frac{0.22}{\omega_{pe,max}} \sim 0.004 \text{ to } 0.02 \text{ fs} \]
No resistivity: GOL and pure two-fluid agree closely

$$\Delta t = \frac{0.22}{\omega_{pe,max}}$$

Log ion density, 333 fs

Log electron density, 333 fs
With no resistivity, GOL and pure two-fluid agree closely

- Black = total energy
- Red, brown = energy in electric, magnetic fields
- Blue = thermal (dashed), kinetic (solid) electron energy
- Green = thermal (dashed), kinetic (solid) ion energy

\[ \Delta t = \frac{0.22}{\omega_{pe, \text{max}}} \]

Pure two-fluid, \( \eta = 0 \)

GOL two-fluid, \( \eta = 0 \)

Total energy, pure two-fluid

Total energy, GOL two-fluid
Resistivity reduces ion ablation

Log ion density, 500 fs

No resistivity

Pure two-fluid

GOL two-fluid

Resistivity
With resistivity, GOL two-fluid shows faster convergence in terms of ion dynamics.

Log ion density, 666 fs

\[ \Delta t = \frac{1}{\omega_{pe,\text{max}}} \]

\[ \Delta t = \frac{0.22}{\omega_{pe,\text{max}}} \]
Electron current loops generate dipolar $B_z$

Out-of-plane magnetic field superimposed with contours of cycle-averaged electron velocity

GOL two-fluid

$\Delta t = \frac{0.22}{\omega_{pe,\text{max}}}$

333 fs  500 fs  666 fs
Conclusions thus far

• Relativistic PERSEUS reproduces expected non-relativistic results.
• Validation against PIC simulation: laser-plasma penetration
  – Penetration ahead of laser by relativistic electron jets with return current and dipolar B-field.
  – Possible mechanisms:
    1. Laser penetration of supercritical plasma via relativistic induced transparency
    2. Ions partially neutralize fields generated by electrons, enabling further penetration (jets have opening angle ~ $v_{iy}/v_{ex}$).
• Laser-foil interaction
  – Compared with pure two-fluid, GOL two-fluid shows better convergence of ion dynamics.
  – Results have sensitivity to resistivity implementation.
  – GOL two-fluid is preferable for modeling dense plasmas.
• Future work
  – Hybrid X-pinches, power feed loss in electrode surface plasmas
  – Relativistic MRT, astrophysical applications